Beta function

The Beta function is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \text{wher, } m>0,\ n>0$$

Put $x = 1-y$
$dx = -dy$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}(\cancel{1}-\cancel{1}+y)^{n-1}(-dy)$$

$$= -\int_1^0 (1-y)^{m-1}(y)^{n-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n,m)$$

$$\Rightarrow \boxed{\beta(m,n) = \beta(n,m)}$$

Put $x = \sin^2\theta$
$dx = 2\sin\theta\cos\theta\,d\theta$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}\cdot(\cos^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= \cancel{\left(2\int_0^{\pi/2} \sin^{2m}\theta\right)}$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi} \sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\,d\theta}$$

which is another form of $\beta(m,n)$

$$\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^{\infty} \frac{x^{n-1}\,dx}{(1+x)^{m+n}}$$

Gamma function (Γ)

The gamma function is defined as.

Finding value of gamma 1.

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx} \quad (n>0$$

Put $n = 1$.

$$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1}\, dx = \int_0^\infty e^{-x} dx$$

$$= \left| \frac{e^{-x}}{-1} \right|_0^\infty = \left| \frac{-1}{e^{x}} \right|_0^\infty = \left(\frac{-1}{e^\infty} + \frac{1}{e^0} \right)$$

This is power of e

$$= \boxed{\Gamma(1) = 1}$$

* Reduction formula for $\Gamma(n)$

$$\Gamma(n+1) = \int_0^\infty \underset{\text{II}}{e^{-x}} \underset{\text{I}}{x^n}\, dx \quad (\text{using ILATE})$$

$$= x^n \int_0^\infty e^{-x} dx - \int n x^{n-1} \frac{e^{-x}}{-1} dx$$

$$= \left| \cancel{-x^n e^{-x}} \right|_0^\infty + n \int_0^\infty e^{-x} x^{n-1} dx$$

This term is zero.

$-x^n \cdot \frac{1}{e^x}$ due to this

$$= n\,\Gamma n$$

Hence, Proved

$$\boxed{\Gamma(n+1) = n\,\Gamma n}$$

* Value of Γn in terms of factorial:

$$\Gamma(2) = 1 \times \Gamma 1 = \underline{|1}$$

$$\Gamma(3) = 2 \times \Gamma 2 = \underline{|2}$$

$$\boxed{\Gamma n = \underline{|n-1}}$$

'n' = +ve Integer.

Value of $\Gamma\frac{1}{2} = \sqrt{\pi} = 1.772$.

* $$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$